Time-Space Trade-Offs for the Longest Common Substring Problem

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The Longest Common Substring Problem

Definition

Problem: Given $T_1, T_2, \ldots, T_m$ of total length n. Compute the longest substring, which appears in at least $2 \le d \le m$ strings.

Example

	1	2	3	4	5	6	7	8	9	10	11	12	13
$T_1 =$	a	g	g	c	t	a	g	c	t	a	c	c	t
$T_2 =$	a	c	a	c	c	t	a	c	c	c	t	a	g
$T_3 =$	a	c	t	a	g	t	a	a	t	g	c	a	t

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$$d = 3 \quad \Rightarrow \quad \mathrm{LCS} = \mathrm{c\ t\ a\ g}$$

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$$d = 3 \quad \Rightarrow \quad \text{LCS} = \text{c t a g}$$

$$d = 2 \quad \Rightarrow \quad \text{LCS} = \text{c t a c c}$$

The Longest Common Substring Problem

A patented solution

US006359574B1

(12) **United States Patent**
Yariv

(10) **Patent No.: US 6,359,574 B1**
(45) **Date of Patent: Mar. 19, 2002**

(54) **METHOD FOR IDENTIFYING LONGEST COMMON SUBSTRINGS**

(75) Inventor: **Shalom Yariv**, Bet Shemesh (IL)

(73) Assignee: **Proxell Systems Ltd.**, Shimshon (IL)

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(21) Appl. No.: **09/821,054**

(22) Filed: **Mar. 30, 2001**

Related U.S. Application Data

(60) Provisional application No. 60/262,654, filed on Jan. 22, 2001.

(51) **Int. Cl.**[7] **H03M 7/00**
(52) **U.S. Cl.** **341/50**; 341/51
(58) **Field of Search** 341/50, 51, 87, 341/107, 86, 63, 95, 106; 708/210, 203

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Primary Examiner—Brian Young
Assistant Examiner—John Nguyen
(74) *Attorney, Agent, or Firm*—AlphaPatent Associates Ltd.; Daniel J. Swirsky

(57) **ABSTRACT**

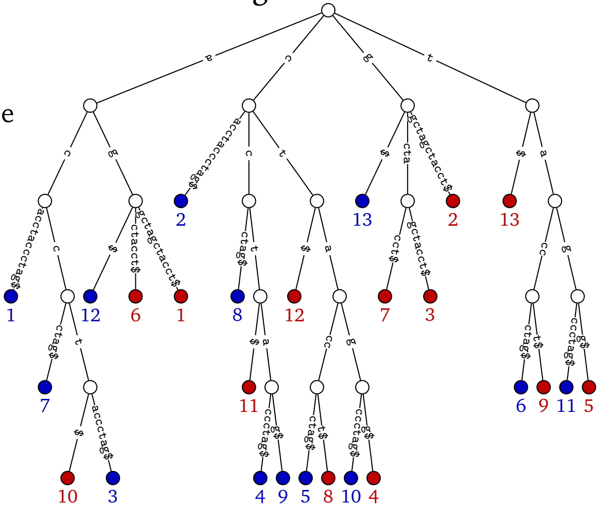
A method for identifying a longest common substring for a string T and a string R, including selecting a registration symbol that appears in both strings R and T, constructing a first relative distance vector R' from the appearance of the registration symbol in the string R, constructing a second relative distance vector T' from the appearance of the registration symbol in the string T, deriving a substring pair R_{CS} and T_{CS} in the strings R and T respectively for each common substring pair R_{CS} and T_{CS} in the vectors R' and T' respectively, and identifying the longest matching of the R_{CS} and T_{CS} substring pairs as the longest common substring for the string T and the string R.

10 Claims, 3 Drawing Sheets

A Textbook Solution

	1	2	3	4	5	6	7	8	9	10	11	12	13
$T_1 =$	a	g	g	c	t	a	g	c	t	a	c	c	t
$T_2 =$	a	c	a	c	c	t	a	c	c	c	t	a	g

Build Generalized Suffix Tree



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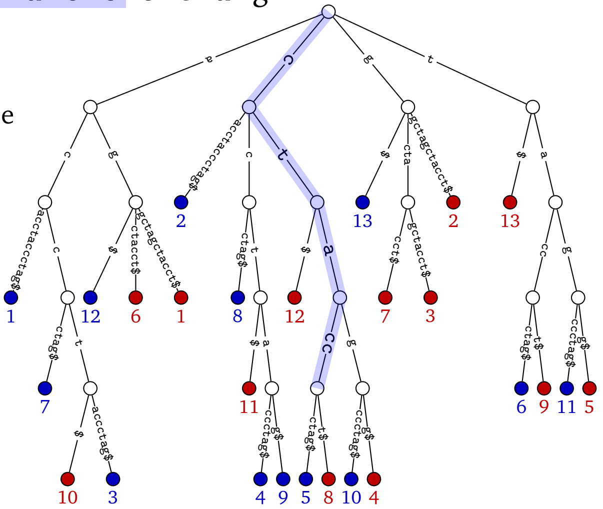

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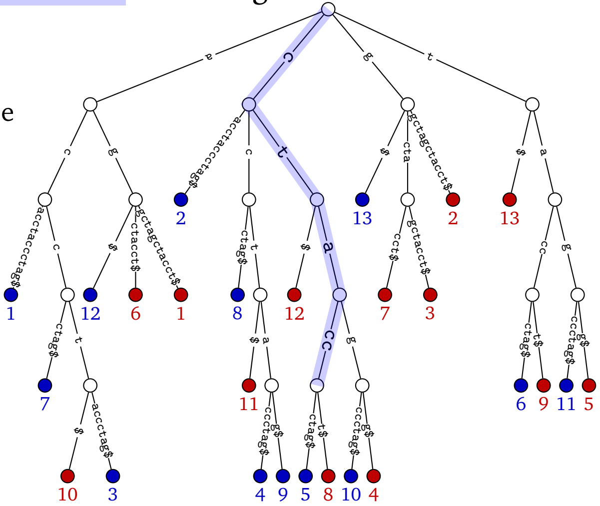
Build Generalized Suffix Tree

Space: $\Theta(n)$

☹



Our Results

Question

Can the LCS problem be solved (deterministically) in $O\left(n^{1-\varepsilon}\right)$ space and $O\left(n^{1+\varepsilon}\right)$ time for $0 \leq \varepsilon \leq 1$?

Our Answer

Yes if $0 \leq \varepsilon \leq \frac{1}{3}$. More precisely,

For two strings ($d = m = 2$), the problem can be solved in:

Time:	$O\left(n^{1+\varepsilon}\right)$	for any $0 < \varepsilon \leq \frac{1}{3}$.
Space:	$O\left(n^{1-\varepsilon}\right)$	

In the general case ($2 \leq d \leq m$), the problem can be solved in:

Time:	$O\left(n^{1+\varepsilon}\log^2 n(d\log^2 n + d^2)\right)$	for any $0 \leq \varepsilon < \frac{1}{3}$.
Space:	$O\left(n^{1-\varepsilon}\right)$	

A Solution for Two Strings

When the LCS is long

Idea: Preprocess a sparse sample of the n suffixes for LCP queries.

$$T = \begin{array}{cccccccccccccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 \\ \mathrm{a} & \mathrm{g} & \mathrm{g} & \mathrm{c} & \mathrm{t} & \mathrm{a} & \mathrm{g} & \mathrm{c} & \mathrm{t} & \mathrm{a} & \mathrm{c} & \mathrm{c} & \mathrm{t} & \$_1 & \mathrm{a} & \mathrm{c} & \mathrm{a} & \mathrm{c} & \mathrm{c} & \mathrm{t} & \mathrm{a} & \mathrm{c} & \mathrm{c} & \mathrm{c} & \mathrm{t} & \mathrm{a} & \mathrm{g} & \$_2 \end{array}$$

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$T =$	a	g	g	c	t	a	g	c	t	a	c	c	t	$\$_1$	a	c	a	c	c	t	a	c	c	c	t	a	g	$\$_2$

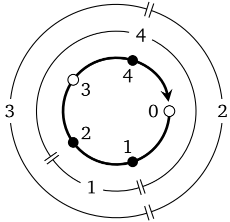
DC_τ DC_τ DC_τ DC_τ DC_τ DC_τ

Difference Covers

A *difference cover modulo* τ is a set of integers $DC_\tau \subseteq \{0, 1, \ldots, \tau - 1\}$ such that for any distance $d \in \{0, 1, \ldots, \tau - 1\}$, DC_τ contains two elements separated by distance d modulo τ.

Ex: The set $DC_\tau = \{1, 2, 4\}$ is a difference cover modulo 5.

d	0	1	2	3	4
i,j	1, 1	2, 1	1, 4	4, 1	1, 2



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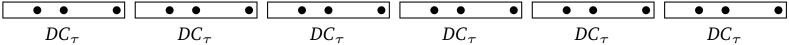

- Number of sampled suffixes: $O\left(\frac{n}{\tau}|DC_\tau|\right) = O\left(\frac{n}{\sqrt{\tau}}\right)$.

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$T =$	a	g	g	c	t	a	g	c	t	a	c	c	t	$\$_1$	a	c	a	c	c	t	a	c	c	c	t	a	g	$\$_2$	
	•	•		•		•	•		•		•	•		•		•	•		•		•	•		•		•	•		•

- Number of sampled suffixes: $O\left(\frac{n}{\tau}|DC_\tau|\right) = O\left(\frac{n}{\sqrt{\tau}}\right)$.
- The LCS is the LCP of two suffixes.

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	•	•		•		•	•		•		•	•		•		•	•		•		•	•		•		•	•		•

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- If $|\text{LCS}| \geq \tau$ one of the first τ characters of the LCS is sampled in both strings.

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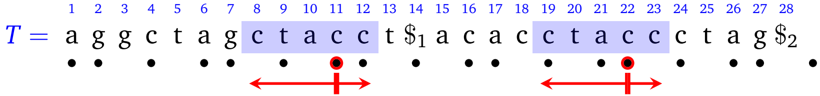

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- Hence the LCS corresponds to a pair (p_1^*, p_2^*) maximizing

$$\mathrm{lcp}\big(RB(p_1), RB(p_2)\big) + \mathrm{lcp}\big(T[p_1..], T[p_2..]\big) - 1$$

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$RB(11) = (\text{g c t a c})^R = \text{c a t c g}$

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A Solution for Two Strings

When the LCS is long

How to compute the pair (p_1^*, p_2^*) faster than $O\left(\frac{n^2}{\tau}\right)$?

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$T =$	a	g	g	c	t	a	g	c	t	a	c	c	t	$\$_1$	a	c	a	c	c	t	a	c	c	c	t	a	g	$\$_2$

$SA_\tau =$ [14 , 21 , 17 , 26 , 6 , 1 , 16 , 22 , 11 , 12 , 19 , 24 , 4 , 27 , 7 , 2 , 9]

$LCP_\tau =$ [0 , 3 , 1 , 2 , 2 , 0 , 1 , 2 , 1 , 2 , 3 , 4 , 0 , 1 , 1 , 0]

$SA_\tau^R =$ [14 , 1 , 17 , 21 , 26 , 6 , 16 , 22 , 11 , 19 , 12 , 24 , 4 , 2 , 27 , 7 , 9]

$LCP_\tau^R =$ [0 , 1 , 1 , 4 , 3 , 0 , 2 , 4 , 1 , 3 , 2 , 1 , 0 , 2 , 4 , 0]

Main observation: $\mathrm{lcp}(T[p_1^*..], T[p_2^*..]) \in [\ell_{max} - \tau + 1; \ell_{max}]$, so we can ignore all pairs with lcp values smaller than $\ell_{max} - \tau + 1$.

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	•	•		•		•	•		•		•	•		•		•	•		•		•	•		•		•	•		•

$SA_\tau = [\,14, 21, 17, 26, 6, 1, 16, 22, 11, 12, 19, 24, 4, 27, 7, 2, 9\,]$

$LCP_\tau = [\,0, 3, 1, 2, 2, 0, 1, 2, 1, 2, 3, 4, 0, 1, 1, 0\,]$

$SA_\tau^R = [\,14, 1, 17, 21, 26, 6, 16, 22, 11, 19, 12, 24, 4, 2, 27, 7, 9\,]$

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	•	•		•		•	•		•		•	•		•		•	•		•		•	•		•		•	•	•

$SA_\tau =$	14	21	17	26	6	1	16	22	11	12	19	24	4	27	7	2	9
$LCP_\tau =$	0	3	1	2	2	0	1	2	1	2	3	4	0	1	1	0	
$SA_\tau^R =$	14	1	17	21	26	6	16	22	11	19	12	24	4	2	27	7	9
$LCP_\tau^R =$	0	1	1	4	3	0	2	4	1	3	2	1	0	2	4	0	

Main observation: $\mathrm{lcp}(T[p_1^*..], T[p_2^*..]) \in [\ell_{max} - \tau + 1; \ell_{max}]$, so we can ignore all pairs with lcp values smaller than $\ell_{max} - \tau + 1$.

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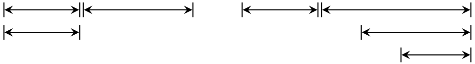
	•	•		•		•	•		•		•	•		•		•	•		•		•	•		•		•	•		•



$$SA_\tau = [\,14, 21, 17, 26, 6, 1, 16, 22, 11, 12, 19, 24, 4, 27, 7, 2, 9\,]$$

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Main observation: $\mathrm{lcp}(T[p_1^*..], T[p_2^*..]) \in [\ell_{max} - \tau + 1; \ell_{max}]$, so we can ignore all pairs with lcp values smaller than $\ell_{max} - \tau + 1$.

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$$
\begin{array}{rcccccccccccccccccccccccccccc}
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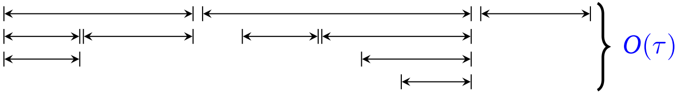
T = & \mathrm{a} & \mathrm{g} & \mathrm{g} & \mathrm{c} & \mathrm{t} & \mathrm{a} & \mathrm{g} & \mathrm{c} & \mathrm{t} & \mathrm{a} & \mathrm{c} & \mathrm{c} & \mathrm{t} & \$_1 & \mathrm{a} & \mathrm{c} & \mathrm{a} & \mathrm{c} & \mathrm{c} & \mathrm{t} & \mathrm{a} & \mathrm{c} & \mathrm{c} & \mathrm{c} & \mathrm{t} & \mathrm{a} & \mathrm{g} & \$_2 \\
 & \bullet & \bullet & & \bullet & & \bullet & \bullet & & \bullet & & \bullet & \bullet & & \bullet & & \bullet & \bullet & & \bullet & & \bullet & \bullet & & \bullet & & \bullet & \bullet & \bullet
\end{array}
$$



$SA_\tau = [\,14, 21, 17, 26, 6, 1, 16, 22, 11, 12, 19, 24, 4, 27, 7, 2, 9\,]$

$LCP_\tau = [\,0, 3, 1, 2, 2, 0, 1, 2, 1, 2, 3, 4, 0, 1, 1, 0\,]$

Analysis (sketch): $O(\tau)$ rounds each using $O(n/\sqrt{\tau})$ time and space:

Time:	$O\left(n\sqrt{\tau}\right)$
Space:	$O\left(n/\sqrt{\tau}\right)$

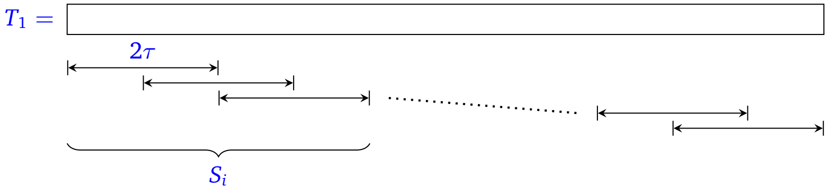
$\xrightarrow{\tau = n^{2\varepsilon}}$

Time:	$O\left(n^{1+\varepsilon}\right)$
Space:	$O\left(n^{1-\varepsilon}\right)$

$0 < \varepsilon \leq \frac{1}{2}$.

A Solution for Two Strings

When the LCS is shorter than τ



- The LCS is a substring of one of the strings of length 2τ.
- Build the generalized suffix tree for a batch S_i of strings of total length $O(\frac{n}{\sqrt{\tau}})$.
- Traverse the suffix tree with T_2 in $O(n)$ time to find the node of greatest string depth.
- Repeat for all $O(\sqrt{\tau})$ batches.

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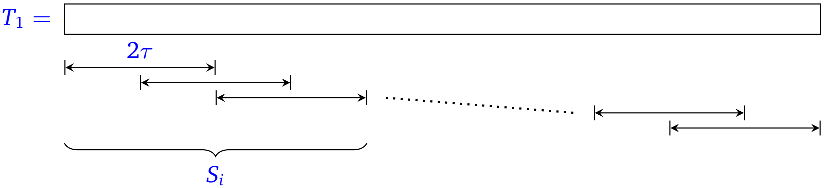

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Time:	$O\left(n\sqrt{\tau}\right)$
Space:	$O\left(n/\sqrt{\tau}\right)$

$\xrightarrow{\tau = n^{2\varepsilon}}$

Time:	$O\left(n^{1+\varepsilon}\right)$
Space:	$O\left(n^{1-\varepsilon}\right)$

$0 \leq \varepsilon \leq \frac{1}{3}$.

$\tau = O(n/\sqrt{\tau})$

Conclusion

Results

For two strings ($d = m = 2$), the LCS problem can be solved in:

Time:	$O\left(n^{1+\varepsilon}\right)$	for any $0 < \varepsilon \leq \frac{1}{3}$.
Space:	$O\left(n^{1-\varepsilon}\right)$	

In the general case ($2 \leq d \leq m$), the LCS problem can be solved in:

Time:	$O\left(n^{1+\varepsilon} \log^2 n (d \log^2 n + d^2)\right)$	for any $0 \leq \varepsilon < \frac{1}{3}$.
Space:	$O\left(n^{1-\varepsilon}\right)$	

Open Problems

Can the generalized solution be improved? Can the trade-off interval of our solutions be extended to $0 \leq \varepsilon \leq \frac{1}{2}$? Can the problem be solved in $O(n^{1+\varepsilon})$ time and $O(n^{1-\varepsilon})$ space for any $0 \leq \varepsilon \leq 1$?